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# Asymptotic Capacity of Multi-User MIMO Correlated Channels

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**Abstract**—This paper introduces a new formula to derive explicit capacity expressions of a class of communication schemes, including single-cell multi-user MIMO and multi-cell point-to-point MIMO, when the wireless channels have separable variance profiles and the system dimensions grow large. As an introductory example, we study point-to-point MIMO channels with multi-cell interference, in downlink. In this setting, we provide new asymptotic capacity expressions when single-user decoding or MMSE decoding are used. Simulations are shown to corroborate the theoretical claims, even when the number of transmit/receive antennas is not very large.

## I. INTRODUCTION

In the last years, while mobile networks were expected to run out of power and frequency resources, Foschini [4] and Telatar [5] introduced the notion of MIMO (multiple input multiple output) systems and predicted a growth of capacity performance of  $\min(n_R, n_T)$  times the single antenna capacity for an  $n_T$ -antenna transmitter and an  $n_R$ -antenna receiver. However, this tremendous multiplexing gain can only be provided for large SINR (signal-to-interference plus noise ratio) and without signal correlation. In case of correlation due to antenna spacing or to poorly scattering environments, these results are still an open issue. In present multi-cell wireless mobile networks, neither base stations nor users cooperate; this leaves the device manufacturers with the dilemma of increasing the signal processing capabilities of the transmit/receive units to result into non significant throughput gains when adjacent cells interfere one another. Moreover, due to limited computational constraints, suboptimal linear techniques such as MMSE (minimum mean square error) decoding are used at the receiver [6], in place of optimal single-user decoders.

In this work, we derive the channel capacity of MMSE receivers against optimal single-user decoders in multi-cell networks, when the number of antennas at the transmitters and receivers is large. The capacity here is defined as the supremum of the achievable rates between a base station and a specific user (in uplink or in downlink) interfered by other cells. We model all transmission channels by the well-spread Kronecker model [7]. Few major contributions propose to study the capacity performance of point-to-point communications with interference. In [11], the authors carry out the performance analysis of TDMA-based networks with inter-cell interference. In [12], a random matrix approach is

used to study large CDMA-based networks with inter-cell interference. In the MIMO context, [8] provides an analytic solution to our problem, using replica methods [9]. These methods are however tedious since they require heavy combinatorial calculus. We propose in the following a more direct approach, based on analytical tools of random matrix theory [10]. In particular, we introduce a new theorem, related to the Stieltjes transform of a specific class of random matrices, which generalizes a similar result in [1].

Although this specific work is dedicated to the study of point-to-point MIMO systems with multi-cell interference, the method we introduce covers a larger class of problems, in which channel capacities express as the log determinant of a sum of Gram matrices  $\mathbf{X}_i \mathbf{X}_i^H$ , where  $\mathbf{X}_i$  is a large matrix modelled as Kronecker. For example, aside from uplink/downlink multi-cell single-user MIMO, this method encompasses single-cell multi-user MIMO communications in the uplink, evaluation of the capacity region of multiple access channels and dirty paper coding in broadcast channels [14] etc.

The remainder of this work is structured as follows: in Section II, we provide mathematical preliminaries and we introduce a new theorem, for which we provide a sketch of the proof. In Section III, we introduce the system model. In Section IV, the point-to-point capacity of the channel between a base station and a user, interfered by other cells, is derived when optimal single-user decoding or MMSE decoding are performed at the receiver. In Section V, we provide simulation results of the previously derived theoretical formulas. Finally, in Section VI, we give our conclusions.

*Notation:* In the following, boldface lower-case symbols represent vectors, capital boldface characters denote matrices ( $\mathbf{I}_N$  is the  $N \times N$  identity matrix).  $X_{ij}$  denotes the  $(i, j)$  entry of  $\mathbf{X}$ . The Hermitian transpose is denoted  $(\cdot)^H$ . The operators  $\text{tr } \mathbf{X}$ ,  $|\mathbf{X}|$  and  $\|\mathbf{X}\|$  represent the trace, determinant and spectral norm of matrix  $\mathbf{X}$ , respectively. The symbol  $\mathbb{E}[\cdot]$  denotes expectation. The notation  $F^{\mathbf{Y}}$  stands for the empirical distribution of the eigenvalues of the Hermitian matrix  $\mathbf{Y}$ .

## II. MATHEMATICAL PRELIMINARIES

Part of this work is dedicated to the introduction of a novel theorem, from which the multi-cell downlink and uplink capacities will be given compact expressions. This theorem gen-

eralizes Silverstein and Bai's formula [1] to random matrices with separable variance profiles, i.e. following the Kronecker model, and unfolds as follows,

**Theorem 1:** (Stieltjes Transform) Let  $K, N \in \mathbb{N}$  be some positive integers. Let

$$\mathbf{B}_N = \sum_{k=1}^K \mathbf{R}_k^{\frac{1}{2}} \mathbf{X}_k \mathbf{T}_k \mathbf{X}_k^H \mathbf{R}_k^{\frac{1}{2}} \quad (1)$$

be an  $N \times N$  matrix with the following hypothesis for all  $k \in \{1, \dots, K\}$ ,

- 1)  $\mathbf{T}_k$  is  $n_k \times n_k$  Hermitian nonnegative definite,  $n_k \in \mathbb{N}^*$ ,
- 2)  $\mathbf{R}_k^{\frac{1}{2}}$  is the  $N \times N$  Hermitian nonnegative definite square root of the nonnegative definite matrix  $\mathbf{R}_k$ ,
- 3) The sequences  $\{F^{\mathbf{T}_k}\}_{n_k \geq 1}$  and  $\{F^{\mathbf{R}_k}\}_{N \geq 1}$  are tight, i.e. for all  $\varepsilon > 0$ , there exists  $M_0 > 0$  such that  $M > M_0$  implies  $F^{\mathbf{T}_k}([M, \infty)) < \varepsilon$  and  $F^{\mathbf{R}_k}([M, \infty)) < \varepsilon$  for all  $n_k, N$ .
- 4)  $\mathbf{X}_k$  is  $N \times n_k$  with i.i.d. complex Gaussian entries with variance  $1/n_k$ .

For  $k \in \{1, \dots, K\}$ , let  $c_k = n_k/N$ . Also denote, for  $z \in \mathbb{C} \setminus \mathbb{R}^+$ ,  $m_N(z) = \frac{1}{N} (\mathbf{B}_N - z\mathbf{I}_N)^{-1}$ . Then, as all  $n_k$  and  $N$  grow large (while  $K$  is fixed), with ratio  $c_k$

$$m_N(z) - m_N^{(0)}(z) \xrightarrow{\text{a.s.}} 0 \quad (2)$$

where

$$m_N^{(0)}(z) = \frac{1}{N} \text{tr} \left( \sum_{k=1}^K \int \frac{\tau_k dF^{\mathbf{T}_k}(\tau_k)}{1 + \frac{\tau_k}{c_k} e_k(z)} \mathbf{R}_k - z\mathbf{I}_N \right)^{-1} \quad (3)$$

and the set of functions  $\{e_i(z)\}$ ,  $i \in \{1, \dots, K\}$ , form the unique solution to the  $K$  equations

$$e_i(z) = \frac{1}{N} \text{tr} \mathbf{R}_i \left( \sum_{k=1}^K \int \frac{\tau dF^{\mathbf{T}_k}(\tau_k)}{1 + \frac{\tau_k}{c_k} e_k(z)} \mathbf{R}_k - z\mathbf{I}_N \right)^{-1} \quad (4)$$

such that  $\text{sgn}(\Im[e_i(z)]) = \text{sgn}(\Im[z])$ .

The function  $m_N(z)$  is the Stieltjes transform [10] of the random variable with cumulative distribution function  $F^{\mathbf{B}_N}$ . The complete proof of a more general expression of this theorem is given in an extended version of the present article [2]. In the following, we give a sketch of the essential steps of the proof

*Proof:* First note that, when  $K = 1$  and, for all  $i$ ,  $\mathbf{R}_i = \mathbf{I}_N$ , the theorem is already known from [1]. We consider here  $K = 1$  (and drop the useless indexes), the general case being a trivial extension, see [2]. Also, we assume here  $\mathbf{T}$  diagonal, which does not restrict generality since the Gaussian matrix  $\mathbf{X}$  is unitarily invariant.

- A first truncation and centralization step makes it possible to bound the entries of the random matrix  $\mathbf{X}$  and the entries of  $\mathbf{R}$ ,  $\mathbf{T}$  to  $\|\mathbf{X}\| \leq k \log(N)$ ,  $k > 2$ ,  $\|\mathbf{R}\| \leq \log(N)$ ,  $\|\mathbf{T}\| \leq \log(N)$ . It is shown first that these truncations and centralizations do not restrict the generality of the final result. Now deterministic bounds can then be used.
- Denote  $\mathbf{D} = -z\mathbf{I}_N - zp(z)\mathbf{R}$ , with  $p(z) = -1/(nz) \sum_{j=1}^n \frac{\tau_j}{1 + \tau_j e(z)/c}$ ,  $\{\tau_i\}$  being the eigenvalues of

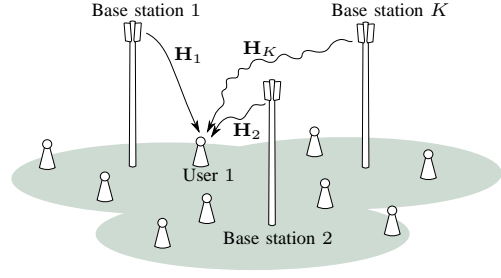


Fig. 1. Downlink multi-cell scenario

$\mathbf{T}$ ,  $e(z) = (1/N) \text{tr} \mathbf{T}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}$  and  $z \in \mathbb{C}^+$ . From the resolvent identity (Equation (4.1) in [1]) and the matrix inversion lemma (Equation (2.2) in [1]),  $(1/N) \text{tr} \mathbf{D}^{-1} - m_N(z) = (1/n) \sum_{j=1}^n \tau_j d_j$  with

$$d_j = \frac{\mathbf{x}_j^H \mathbf{R}^{\frac{1}{2}} (\mathbf{B}_{(j)} - z\mathbf{I}_N)^{-1} \mathbf{D}^{-1} \mathbf{R}^{\frac{1}{2}} \mathbf{x}_j}{1 + \tau_j \mathbf{y}_j^H (\mathbf{B}_{(j)} - z\mathbf{I}_N)^{-1} \mathbf{y}_j} \quad (5)$$

$$- \frac{(1/N) \text{tr} \mathbf{R}(\mathbf{B}_N - z\mathbf{I}_N)^{-1} \mathbf{D}^{-1}}{1 + \tau_i e(z)/c} \quad (6)$$

where  $\mathbf{x}_j$  is the  $j^{\text{th}}$  column of  $\mathbf{X}$ ,  $\mathbf{y}_j$  the  $j^{\text{th}}$  column of  $\mathbf{R}^{\frac{1}{2}} \mathbf{X}$  and  $\mathbf{B}_{(j)}$  is  $\mathbf{B}_N$  with  $j^{\text{th}}$  column removed. Observing that  $d_j \rightarrow 0$  when  $N \rightarrow \infty$ , we have  $(1/N) \text{tr} \mathbf{D}^{-1} - m_N(z) \rightarrow 0$  and similarly  $(1/N) \text{tr} \mathbf{D}^{-1} \mathbf{R} - e(z) \rightarrow 0$ .

- The rest of the proof consists in proving the existence and uniqueness of a solution to Equation (4), from which we have the existence of a unique solution  $m_N^{(0)}$  in Equation (3). It is then proven that, with the initial hypothesis for  $\mathbf{X}$ ,  $\mathbf{T}$  and  $\mathbf{R}$ ,  $m_N(z) - m_N^{(0)}(z) \xrightarrow{\text{a.s.}} 0$ .
- Applying finally Vitali's theorem to the analytic function  $m_N^{(0)}$ , the theorem is shown to hold for all  $z \in \mathbb{C} \setminus \mathbb{R}^+$ .

**Remark 1:** This theorem allows us to derive Stieltjes transforms of large matrices *independently* of the realization of the  $\mathbf{X}_k$  matrices. In wireless communications, this provides a characterization of a multi-user or multi-cell communication based only on the transmit and receive correlations  $\mathbf{R}_k$  and  $\mathbf{T}_k$ . This further helps to estimate channel capacity thanks to the Shannon transform,

**Theorem 2:** (Shannon Transform) Let  $\mathbf{B}_N$  be a random Hermitian matrix as defined in Theorem 1 with the additional assumption that there exists  $M > 0$ , such that, for all  $N, n_k$ ,  $\max(\|\mathbf{T}_k\|, \|\mathbf{R}_k\|) < M$ , and let  $x > 0$ . Then, for large  $N, n_k$ ,  $\mathcal{V}(x) - \mathcal{V}^{(0)}(x) \xrightarrow{\text{a.s.}} 0$ , where

$$\mathcal{V}(x) = \int \log_2 \left( 1 + \frac{b}{x} \right) dF^{\mathbf{B}_N}(b) \quad (7)$$

and

$$\mathcal{V}^{(0)}(x) = \int_x^{+\infty} \left( \frac{1}{w} - m_N^{(0)}(-w) \right) dw \quad (8)$$

A proof of this result is provided in [2].

### III. SYSTEM MODEL

In this work we derive capacity expressions of wireless channels between a multi-antenna transmitter and a multi-antenna receiver, the latter of which is interfered by several multi-antenna transmitters. This scheme is well-suited to multi-cell wireless networks with orthogonal intra-cell and interfering inter-cell transmissions, both in downlink and in uplink. The following scenarios encompass in particular

- multi-cell uplink: the base station of a cell indexed by  $i \in \{1, \dots, K\}$  receives data from one user in this cell<sup>1</sup> and is interfered by  $K - 1$  users transmitting on the same physical resource from remote cells indexed by  $j \in \{1, \dots, K\}, j \neq i$ .
- multi-cell downlink: the user being allocated a given time/frequency resource in a cell indexed by  $i \in \{1, \dots, K\}$  receives data from its dedicated base-station and is interfered by  $K - 1$  base stations in neighboring cells indexed by  $j \in \{1, \dots, K\}, j \neq i$ . This situation is depicted in Figure 1.

In the following, in order not to confuse both scenarios, only the downlink scheme is considered. However, one must keep in mind that the provided results can easily be adapted to the uplink case.

Consider a wireless mobile network with  $K \geq 1$  cells indexed from 1 to  $K$ , controlled by *non-physically connected* base stations. On a particular time/frequency resource, each base station serves only one user; therefore the base station and the user of cell  $j$  will also be indexed by  $j$ . Without loss of generality, we focus our attention on user 1, equipped with  $n_R \gg K$  antennas and hereafter referred to as *the user* or *the receiver*. Every base station  $j \in \{1, \dots, K\}$  is equipped with  $n_{T_j} \gg K$  antennas. We additionally denote  $c_j = n_{T_j}/n_R$ .

Denote  $\mathbf{s}_j \in \mathbb{C}^{n_{T_j}}$ ,  $\mathbb{E}[\mathbf{s}_j \mathbf{s}_j^H] = \mathbf{I}_{n_{T_j}}$ , the signal transmitted by base station  $j$ ,  $\mathbf{y} \in \mathbb{C}^{n_R}$  the signal received by the user and  $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_{n_R})$  the noise vector received by the user. The fading MIMO channel between base station  $j$  and the user is denoted  $\mathbf{H}_j \in \mathbb{C}^{n_R \times n_{T_j}}$ . Moreover we assume that  $\mathbf{H}_j$  has a separable variance profile, i.e. can be decomposed as

$$\mathbf{H}_j = \mathbf{R}_j^{\frac{1}{2}} \mathbf{X}_j \mathbf{T}_j^{\frac{1}{2}} \quad (9)$$

with  $\mathbf{R}_j \in \mathbb{C}^{n_R \times n_R}$  the (Hermitian) correlation matrix at the receiver with respect to the channel  $\mathbf{H}_j$ ,  $\mathbf{T}_j \in \mathbb{C}^{n_{T_j} \times n_{T_j}}$  the correlation matrix at transmitter  $j$  and  $\mathbf{X}_j \in \mathbb{C}^{n_R \times n_{T_j}}$  a random matrix with Gaussian independent entries of variance  $1/n_{T_j}$ .

*Remark 2:* Note that in this model, and contrary to what is often assumed,  $\mathbf{R}_j$ , the correlation matrix at the receiver, depends on  $j$ . In the uplink scenario, this assumption is of particular relevance in the sense that base stations are usually placed in areas clear of scatterers. In these circumstances, the solid angle from which the signals from user  $j$  originate influences the signal correlation at the receive antenna array.

<sup>1</sup>this user is allocated a given time/frequency resource, which is orthogonal to time/frequency resources of the other users in the cell; e.g. the multi-access protocol is OFDMA.

Hence the dependence of the receive correlation matrices on  $j$ . Note moreover that, in this model, the transmit power assumption  $\mathbb{E}[\mathbf{s}_j \mathbf{s}_j^H] = \mathbf{I}_{n_{T_j}}$  is not restrictive in the sense that the transmit power correlation of base station  $j$  can be included into the matrix  $\mathbf{T}_j$ . However, the Kronecker model has two major drawbacks: (i) the inner matrix  $\mathbf{X}_j$  implicitly assumes a high density of scatterers<sup>2</sup> in the communication link and (ii) the correlations on both sides must be inter-independent and independent of the realizations of  $\mathbf{X}_j$ , which is inaccurate to some extent.

With the assumptions above, the communication model unfolds

$$\mathbf{y} = \mathbf{H}_1 \mathbf{s}_1 + \sum_{j=2}^K \mathbf{H}_j \mathbf{s}_j + \mathbf{n} \quad (10)$$

where  $\mathbf{s}_1$  is the useful signal (from base station 1) and  $\mathbf{s}_j$ ,  $j \geq 2$ , constitute interfering signals.

### IV. MULTI-CELL MIMO CAPACITY

#### A. Optimal Single-User Decoding

If the receiving user considers the signals from the  $K - 1$  interfering transmitters as correlated Gaussian noise and knows the value of the SNR (signal-to-noise ratio)  $\sigma^{-2}$ , then base station 1 can transmit with arbitrarily low decoding error at a rate per-receive antenna  $C_{\text{opt}}(\sigma^2)$  given by [3]

$$C_{\text{opt}}(\sigma^2) = \frac{1}{n_R} \log_2 |\mathbf{I}_{n_R} + \frac{1}{\sigma^2} \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H| - \frac{1}{n_R} \log_2 |\mathbf{I}_{n_R} + \frac{1}{\sigma^2} \sum_{j=2}^K \mathbf{H}_j \mathbf{H}_j^H| \quad (11)$$

Assume that  $n_R$  and the  $n_{T_i}$ ,  $i \in \{1, \dots, K\}$ , are large compared to  $K$  and such that no eigenvalue of  $\mathbf{R}_i$  or  $\mathbf{T}_i$  is too large. As in Theorem 1, we define the function  $m^{(0)}$  as the asymptotic Stieltjes transform of  $\mathbf{B}_N = \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H$ ,

$$m^{(0)}(z) = \frac{1}{n_R} \text{tr} \left( \sum_{j=1}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} e_j(z)} \mathbf{R}_j - z \mathbf{I}_{n_R} \right)^{-1} \quad (12)$$

where, for all  $i \in \{1, \dots, K\}$ ,  $e_i(z)$  is solution of the fixed-point equation

$$e_i(z) = \frac{1}{n_R} \text{tr} \mathbf{R}_i \left( \sum_{j=1}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} e_j(z)} \mathbf{R}_j - z \mathbf{I}_{n_R} \right)^{-1} \quad (13)$$

From Theorem 2, applied to  $\mathbf{B}_N = \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H$ , we then have approximately

$$\frac{1}{n_R} \log_2 |\mathbf{I}_{n_R} + \frac{1}{\sigma^2} \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H| = \int_{\sigma^2}^{+\infty} \left( \frac{1}{w} - m^{(0)}(-w) \right) dw \quad (14)$$

<sup>2</sup>the number and distance between scatterers must be of the same order as the number and distance between the transmit and receive antennas.

A similar result is obtained for the second right-hand side term of Equation (11). The per-receive antenna capacity  $C_{\text{opt}}(\sigma^2)$  is therefore well approximated, for large number of antennas, by

$$C_{\text{opt}}(\sigma^2) = \quad (15)$$

$$\frac{1}{n_R} \int_{\sigma^2}^{+\infty} \left[ \text{tr} \left( \sum_{j=1}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} e_j(w)} \mathbf{R}_j - w \mathbf{I}_{n_R} \right)^{-1} \right. \\ \left. - \text{tr} \left( \sum_{j=2}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} f_j(w)} \mathbf{R}_j - w \mathbf{I}_{n_R} \right)^{-1} \right] dw \quad (16)$$

where,  $e_i, i \in \{1, \dots, K\}$ , and  $f_i, i \in \{2, \dots, K\}$ , verify

$$e_i(z) = \frac{1}{n_R} \text{tr} \mathbf{R}_i \left( \sum_{j=1}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} e_j(z)} \mathbf{R}_j - z \mathbf{I}_{n_R} \right)^{-1} \quad (17)$$

$$f_i(z) = \frac{1}{n_R} \text{tr} \mathbf{R}_i \left( \sum_{j=2}^K \int \frac{t_j dF^{\mathbf{T}_j}(t_j)}{1 + \frac{t_j}{c_j} f_j(z)} \mathbf{R}_j - z \mathbf{I}_{n_R} \right)^{-1} \quad (18)$$

### B. MMSE Decoder

Achieving  $C_{\text{opt}}$  requires non-linear processing at the receiver, such as MMSE successive interference cancellation. A suboptimal linear technique, the MMSE decoder, is often used instead. The communication model in this case reads

$$\mathbf{y} = \left( \sum_{j=1}^k \mathbf{H}_j \mathbf{H}_j^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \mathbf{H}_1^H \left( \sum_{j=1}^k \mathbf{H}_j \mathbf{s}_j + \mathbf{n} \right) \quad (19)$$

and each entry of  $\mathbf{y}$  will be processed individually.

This technique makes it possible to transmit data reliably at any rate inferior to the per-antenna MMSE capacity  $C_{\text{MMSE}}$ ,

$$C_{\text{MMSE}}(\sigma^2) = \frac{1}{n_R} \sum_{i=1}^{n_{T_1}} \log_2(1 + \gamma_i) \quad (20)$$

where, denoting  $\mathbf{h}_j \in \mathbb{C}^{n_{T_j}}$  the  $j^{\text{th}}$  column of  $\mathbf{H}_1$  and  $\mathbf{R}_1^{\frac{1}{2}} \mathbf{x}_j = \mathbf{h}_j$ , the individual SINR  $\gamma_i$ 's express as

$$\gamma_i = \frac{\mathbf{h}_i^H \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \mathbf{h}_i}{1 - \mathbf{h}_i^H \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \mathbf{h}_i} \quad (21)$$

$$= \mathbf{h}_i^H \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H - \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \mathbf{h}_i \quad (22)$$

$$= \mathbf{x}_i^H \mathbf{R}_i^{\frac{1}{2}} \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H - \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \mathbf{R}_i^{\frac{1}{2}} \mathbf{x}_i \quad (23)$$

where Equation (22) comes from a direct application of the matrix inversion lemma. With these notations,  $\mathbf{x}_i$  has i.i.d. complex Gaussian entries with variance  $T_{1_{ii}}/n_{T_i}$  and the inner

matrix of the right-hand side of (23) is independent of  $\mathbf{x}_i$  (since the entries of  $\mathbf{H}_1 \mathbf{H}_1^H - \mathbf{h}_i \mathbf{h}_i^H$  are independent of the entries  $\mathbf{h}_i$ ). Applying Lemma 3.1 of [1], for  $n_{T_i}$  large, approximately

$$\gamma_i = \frac{T_{1_{ii}}}{n_{T_1}} \text{tr} \mathbf{R}_1 \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H - \mathbf{h}_i \mathbf{h}_i^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \quad (24)$$

From Lemma 2.1 of [13], the rank 1 perturbation  $(-\mathbf{h}_i \mathbf{h}_i^H)$  does not affect asymptotically the trace in (24). And therefore, approximately,

$$\gamma_i = \frac{T_{1_{ii}}}{n_{T_1}} \text{tr} \mathbf{R}_1 \left( \sum_{j=1}^K \mathbf{H}_j \mathbf{H}_j^H + \sigma^2 \mathbf{I}_{n_R} \right)^{-1} \quad (25)$$

Observing that  $e_1(z)$  in Section IV-A corresponds to the normalized trace in Equation (25) (this is shown precisely in the proof of Theorem 1 [2]), we finally have the compact expression for  $C_{\text{MMSE}}$ ,

$$C_{\text{MMSE}}(\sigma^2) = \frac{1}{n_R} \sum_{i=1}^{n_{T_1}} \log_2 \left( 1 + \frac{1}{c_1} T_{1_{ii}} e_1(-\sigma^2) \right) \quad (26)$$

In practice, when no power allocation strategy is applied,  $T_{1_{ii}} = P$  the average power per transmit symbol, and the capacity becomes  $C_{\text{MMSE}} = c_1 \cdot \log_2(1 + \frac{P}{c_1} e_1(-\sigma^2))$ .

## V. SIMULATION AND RESULTS

In the following, we apply the results (15) and (26) to the downlink of a two-cell network. The capacity analyzed here is the achievable rate on the link between base station 1 and the user, the latter of which is interfered by base station 2. The relative power of the signal received from base station 2 is on average  $\Gamma$  times that of base station 1. Both base stations are equipped with linear arrays of  $n_T$  antennas and the user with a linear array of  $n_R$  antennas. The correlation matrices  $\mathbf{T}_i$  at the transmission and  $\mathbf{R}_i$  at the reception,  $i \in \{1, 2\}$ , are modeled thanks to a generalization of Jake's model including solid angles of transmit/receive power, i.e. for instance,

$$T_{i_{ab}} = \int_{\theta_{\min}^{(i)}}^{\theta_{\max}^{(i)}} \exp \left( 2\pi \cdot i \cdot \frac{d_{ab}^{\mathbf{T}_i}}{\lambda} \cos(\theta) \right) d\theta \quad (27)$$

with  $d_{ab}^{\mathbf{T}_i}$  the distances between antennas indexed by  $a, b \in \{1, \dots, n_{T_i}\}$  for transmitter  $i$ ,  $(\theta_{\min}^{(i)}, \theta_{\max}^{(i)})$  the angles over which useful power (i.e. power that will be received by the user) is transmitted, and  $\lambda$  the wavelength.

In Figure 2, we took  $n_R = 16$ ,  $\Gamma = 0.25$  and we consider optimal single-user decoding at the receiver. For every realization of  $\mathbf{T}_i, \mathbf{R}_i$ , 1000 channel realizations are processed to produce the simulated ergodic capacity and compared to the theoretical capacity (15). Those capacities are then averaged over 100 realizations of  $\mathbf{T}_i, \mathbf{R}_i$ , varying in the random choice of  $\theta_{\min}^{(i)}$  and  $\theta_{\max}^{(i)}$  with constraint  $\theta_{\max}^{(i)} - \theta_{\min}^{(i)} = \pi/2$ , while  $d_{ab}^{\mathbf{T}_i} = 10\lambda|a - b|$  at the transmitters,  $d_{ab}^{\mathbf{R}_i} = 2\lambda|a - b|$  at the receiver. The SNR ranges from  $-5$  dB to  $30$  dB, and  $n_T \in \{8, 16\}$ . We observe here that Monte-Carlo simulations perfectly match the capacity obtained from Equation (15).

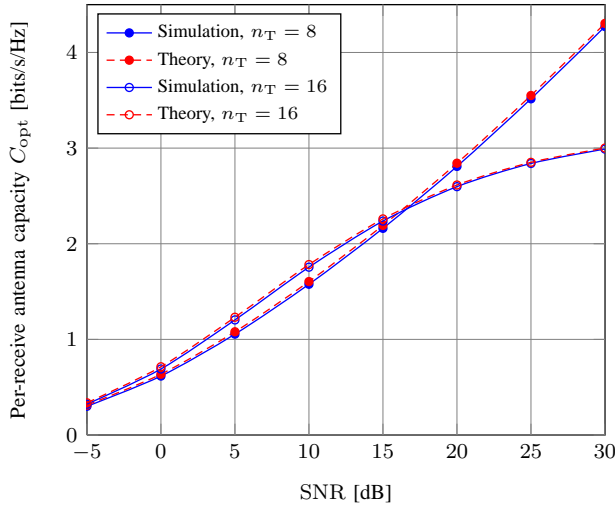


Fig. 2. Capacity of point-to-point MIMO in two-cell uplink, optimal single-user decoding,  $n_R = 16$ ,  $n_T \in \{8, 16\}$ ,  $\Gamma = 25\%$ .

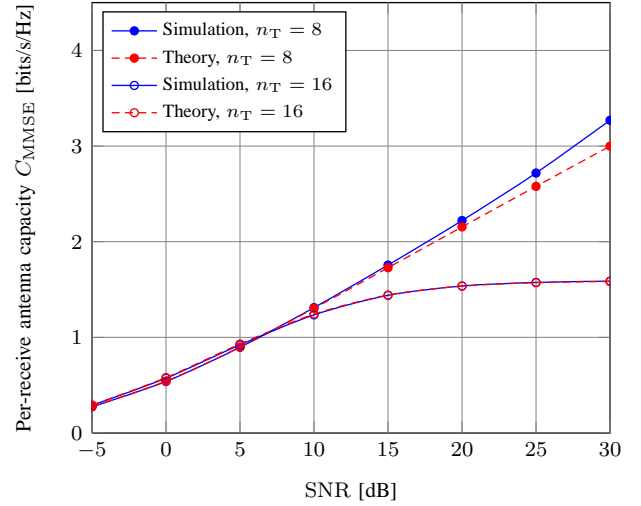


Fig. 3. Capacity of point-to-point MIMO in two-cell uplink, with MMSE decoder,  $n_R = 16$ ,  $n_T \in \{8, 16\}$ ,  $\Gamma = 25\%$ .

In Figure 3, with the same assumptions as previously, we apply MMSE decoding at the receiver. Here, a slight difference is observed in the high SNR regime between theory and practice. This was somehow expected, since the large  $n_R$  approximations in Silverstein's lemmas [1] are very loose for  $\sigma^2$  close to  $\mathbb{R}^-$  in the sense of the Euclidean distance. To cope with this gap, many more antennas must be used. We also observe a significant difference in performance between optimum single-user and linear MMSE decoders, especially in the high SNR region. Therefore, in wireless networks, when interfering cells are treated as correlated Gaussian noise at the cell-edge, i.e. where the interference is maximum, the MMSE decoder provides tremendous performance loss.

## VI. CONCLUSION

In this paper, we introduced a theorem relating the Stieltjes transform of a class of large matrices to deterministic approximates. Based on this formula, we provided compact capacity expressions for the optimal single-user decoder and MMSE decoder in point-to-point MIMO systems with inter-cell interference and random channel matrices with separable variance profile, both in downlink and in uplink. The simulations show perfect match with the theoretical formulas in the low-to-medium SNR region, even if fewer antennas are used at the transmitters and receivers. As for the high SNR region, a large number of antennas must be used to reach an accurate match between theory and Monte-Carlo simulations.

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